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Rank-Order Tournaments

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1 Introduction

There exist numerous types of compensation schemes. In practice, though, not necessarily optimal, hourly and monthly rates most of the time. Questions widely addressed among economists are why do organizations choose one incentive payment scheme over the others and does there even exist an optimal and feasible one? Answers to those question are of a vital importance to economic theory and thus vast majority of papers on this topic are devoted to comparison of different incentive compensation schemes.

One possible classification framework for incentive payment schemes is dividing them into compensations based on absolute and relative performance. The former of the two takes into account absolute level of output produced by an agent, while the latter uses information on relative position of the agent's output, compared to that of other agents. On one extreme of this classification lays individual piece-rate, under which an agent receives a payment linearly dependent on her absolute performance, on the other - rank-order tournament, where the payment to the agent depends on her rank, i.e. her performance relative to other agents. Between the to absolutes are cardinal tournaments or relative piece-rates, a payment structure including some elements of individual piece-rates and ordinal tournaments, i.e. consisting of a base payment proportional to the absolute performance and a bonus or penalty depending on average performance of all agents. All of the three schemes were extensively analyzed and compared at different times in the literature and in the next few paragraphs I will present a brief history of that analysis.

Before the prominent paper by Lazear and Rosen (1981), most of the focus was on analyzing simple linear piece rate schemes (e.g., see Stiglitz (1975), Mirrlees (1976)), under which an agent receives compensation as function of the output she produces, widely used in various contracts, examples here would be a payment to salesman based on the sales he made, a payment to a worker on the plantation linearly dependent on the amount of harvested crop. On the other hand, Lazear and Rosen (1981) and later Holmström (1982), Green and Stokey (1983) and Nalebuff and Stiglitz (1983) analyzed more intriguing rank-order tournaments and compared them to other schemes to find that under certain circumstances the former can be superior to many others. Careful study of rank-order tournaments helps explain why are those at the top of the pyramid (refering to CEOs mainly) are paid so much, why, when only best three of Olympiad athletes are rewarded and others, sometimes being just one hundredths of a second behind, get virtually nothing, so many people are eager to participate?

In their work Lazear and Rosen (1981) examined an ordinal tournament with two agents, both for the case where agents were risk-neutral and the one where they were risk-averse. Authors show that when the agents are riskneutral, there can be designed a tournament which will ensure the same allocation of resources as an individual piece-rate. This equivalence vanishes when agents are assumed to be risk-averse: depending on various parameters, tournament may or may not be superior in terms of allocation to piece rate. Finally, the paper considers a case with heterogeneous workers, yielding some very different results, which are beyond the scope of this discussion. Lazear and Rosen, though, do not generalize results to more than two players, and this issue is addressed in Malcomson (1986), where author solves a game for the continuum of players and most recently in Akerlof and Holden (2012).

Holmström (1982) analyzed various relative compensation schemes, among which a rank-order tournaments. He considered production functions of two types, with multiplicative and additive shocks, and many risk-averse agents. Holmström main finding is that "rank-order tournaments may be informationally quite wasteful f performance levels can be measured cardinally rather than ordinally" (p.335).

Further investigation of the optimality of rank-order tournaments relative to individual compensation schemes is offered by Green and Stokey (1983). In their paper they have a setup very similar to Lazear and Rosen (1981) and Holmström (1982), the main distinctive assumption made is that Green and Stokey allow agents to observe private signals, correlated with this common shock,before they chose their effort levels" (p.3). Authors find that rank-order tournament's superiority in terms of allocation of resources depends largely on the common shock structure and the number of agents in participating in the tournaments: with a common error term or the number of agents being sufficiently large, a tournament is able to eliminate the major source of volatility in production.

Lately, most of the literature on tournaments is related to cardinal version of the scheme (e.g. Marinakis and Tsoulouhas (2013)), i.e. a scheme under which an agent receives a base payment and a bonus or penalty depending on the agent performance relative to the average performance of all agents in the group, as this type of incentive payment scheme is argued by Holmström (1982) to be superior in terms of use of information to the rank-order tournaments. As for ordinal tournaments, the interest of the researches has shifted from pure theory towards empirical work based on experiments. In this regard, Bull, Schotter, and Weigelt (1987) find supportive evidence for the theory outlined in Lazear and Rosen (1981).

In contrast with current research, this paper will be devoted to the analyses of rank-order tournaments per se. I believe that this is the most spectacular of all compensation schemes as so many sports use it. This paper was apparently developed parallel to the work of Akerlof and Holden (2012) and thus one of the objectives of this research will be to generalize results of Lazear and Rosen to a contract for three agents and a contract with N discretely distributed agents. As Bull, Schotter, and Weigelt (1987) mention in their article "a tournament, unlike the piece rate, is a game and so requires strategic, as opposed to simply maximizing, behavior" (p. 3), I will use strategic approach in this paper. The main goal is to see whether the results of Lazear and Rosen (1981) still hold, ones we abolish the assumption of perfect competition and thus drop the zero-profit constraint and increase the number of players to 3 or to N. Due to this difference in assumptions, I will employ backward induction to solve the game and to find optimal value-maximizing prize structure explicitly in ordinal tournament. The main advantage of this work when compared to Akerlof and Holden (2012) is that in their paper Akerlof and Holden try to solve very general form of the problem which requires a lot of assumptions and do not provide exact prizes structure, while this work explicitly finds the optimal prize structure for the tournament with 3 risk-neutral agents and provides some insights for the 3 risk-averse.

The rest of the work is organized as follows: Section 2 describes a case of a contract between a risk-neutral principal and 3 risk-neutral players, in Section 3 a more realistic situation is considered, i.e. players are now assumed to be risk-averse, Section 4 presents an attempt to generalize a model to N agents and describes some difficulties with that, finally, in Section 5 provides a review the work done and presents conclusions.

2 The Rank-Order Tournament with 3 Risk-Neutral Agents

2.a Setup

In this paper only one period is under consideration, during this period a contract is signed between a principal and 3 homogeneous agents. Under the contract agent i produces output x_i according to the production function (used by Marinakis and Tsoulouhas (2013)):

$$x_i = e_i + a + \eta + \epsilon_i \tag{1}$$

where a is agent's ability, equal for all agents, e_i is agent i's effort level, η is a common shock, $\eta \sim N(0, \sigma_\eta^2)$, and ϵ_i is an idiosyncratic shock, $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ *iid*. This production function differs from that used by Lazear and Rosen in two ways: the ability is separated from investment (effort) and the risk is split into common, affecting all agents, for example, economics crisis, which is not diversifiable for the principal, and specific risk to each agent, for example, sickness, which is diversifiable for the principal.

For simplicity of exposition, the price of the output is normalized to 1. Under the contract all of the resulted revenue goes to the principal, while agents are compensated according to their "rank", i.e. depending on the value of their output relative to that of other agents. Here and in all that follows I assume that the principal is risk-neutral and acts as a value-maximizer. In this section agents are assumed to be risk-neutral as well. I will use a utility function for an agent i that is separable in income and cost of effort, following absolute majority of papers on the topic:

$$u(w_i, e_i) = w_i - \frac{e_i^2}{2a} \tag{2}$$

where w_i is a wage received by the agent i and $\frac{e_i^2}{2a}$ is a cost of effort for the agent, decreasing in ability. Under the ordinal tournament the wage rate (prize) received by the agent depends on the rank of x_i, so the precise structure of the compensation scheme for 3 agnets is as follows:

$$w_i = \begin{cases} W_1 & \text{if } x_i > x_j \; \forall i \neq j \\ W_3 & \text{if } x_i < x_j \; \forall i \neq j \\ W_2 & \text{otherwise} \end{cases} \tag{3}$$

where $W_1 \geq W_2 \geq W_3$ or, more specifically let $W_2 = \alpha W_1 + (1-\alpha)W_3$, where $0 \leq \alpha \leq 1$.

All the above information is available both to the principal and the agents. An interesting peculiarity of the rank-order tournaments when compared to other incentive schemes like piece rates is that the model can be viewed as a non-cooperative game: at stage 1 the principal designs a compensation scheme, at stage 2 the agents are offered this compensation scheme, if agents agree to sign a contract, then at stage 3 they choose the level of investment (effort) and at stage 4 the production happens and agents get compensation. Because each agent chooses the level of effort (investment) before the production takes place and after, when the output is observed each agent receives her compensation, this model is similar to Stackelberg leadership model with first-mover advantage for the principal.

Therefore we will use backwards induction: first the principal will calculate each agent's effort level as a function of a prize structure, for this the principal needs to calculate the agent's expected utility first, as the she knows that each agent will choose to sign a contract only if the expected payment gives her maximum expected utility comparing to other offers, i.e. the contract is incentive-compatible. At the same time, the expected payment must also provide the agent with utility level at least equal to her reservation utility (let us assume that it is the utility level an agent enjoys staying out of production), i.e. satisfy individual rationality constraint. Finally, principal maximizes her total expected profits subject to the above incentive compatibility and individual rationality constraints.

The rest of the section will thus reflect the structure of the process of solving the game outlined above.

2.b Probabilities

The expected utility for the agent i is simply the expected wage less the cost of effort:

$$EU = W_1 P(\text{Rank First}) + W_2 P(\text{Rank Second}) + W_3 P(\text{Rank Third}) - \frac{e_i^2}{2a} \tag{4}$$

To find EU we need to find the probability of each payoff, i.e.:

$$P(\text{Rank First}) = P_1 = P(x_i > x_j \; and \; x_i > x_k)$$

Substituting each agent's production function (1) into the above expression and rearranging further, we get:

$$\begin{aligned} P_1 &= P(e_i + a + \eta + \epsilon_i > e_j + a + \eta + \epsilon_j \; and \; e_i + a + \eta + \epsilon_i > e_k + a + \eta + \epsilon_k) = \\ &= P(e_i + \epsilon_i > e_j + \epsilon_j \; and \; e_i + \epsilon_i > e_k + \epsilon_k) = \\ &= P(\epsilon_i - \epsilon_j < e_i - e_j \; and \; \epsilon_i - \epsilon_k < e_i - e_k) \end{aligned}$$

As $cov(\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k) = var(\epsilon_i) = \sigma_\epsilon^2 \neq 0$, random variables $\epsilon_i - \epsilon_j$ and $\epsilon_i - \epsilon_k$ have dependent distributions and thus we can find the probability of ranking first as:

$$\begin{aligned} P_1 &= F_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(e_i - e_j, e_i - e_k) = \\ &= \int_{-\infty}^{e_i - e_j} \int_{-\infty}^{e_i - e_k} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, t) ds dt \end{aligned} \tag{5}$$

where $F_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(.)$ is the joint cdf of $e_i - e_j$ and $e_i - e_k$, $f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(.)$ is its joint pdf. By analogy, the probability of ranking third out of three agents is:

$$\begin{aligned} P(\text{Rank Third}) = P_3 &= P(x_i < x_j \; and \; x_i < x_k) = \\ &= \int_{e_i - e_j}^{\infty} \int_{e_i - e_k}^{\infty} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, t) ds dt \end{aligned} \tag{6}$$

So probability of receiving W_2, in other words, raking second can be found as:

$$P(\text{Rank Second}) = P_2 = 1 - P_1 - P_3 \tag{7}$$

2.c Incentive Compatibility

After calculating the probabilities of each outcome, we can now substitute the expressions (5), (6) and (7) into the expected utility for agent i (4) to get:

$$EU = W_1 P_1 + W_2(1 - P_1 - P_3) + W_3 P_3 - \frac{e_i^2}{2a} \tag{8}$$

The principal want to find the level of effort that the agent will choose treating prize specification as given, so she maximizes the agent's expected utility (8) with respect to e_i. First order condition is:

$$\begin{gathered} \frac{\partial EU}{\partial e_i} = 0 \;\Leftrightarrow \\ W_1 \frac{\partial P_1}{\partial e_i} + W_2(-\frac{\partial P_1}{\partial e_i} - \frac{\partial P_3}{\partial e_i}) + W_3 \frac{\partial P_3}{\partial e_i} - \frac{e_i}{a} = 0 \end{gathered} \tag{9}$$

Differentiating P_1 from (5) with respect to e_i, we can easily find $\frac{\partial P_1}{\partial e_i}$:

$$\begin{aligned} \frac{\partial P_1}{\partial e_i} &= \frac{\partial}{\partial e_i} \left(\int_{-\infty}^{e_i - e_j} ds \left[\int_{-\infty}^{e_i - e_k} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, t) dt \right] \right) = \\ &= \int_{-\infty}^{e_i - e_k} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(e_i - e_k, t) dt + \int_{-\infty}^{e_i - e_j} ds \frac{\partial}{\partial e_i} \left[\int_{-\infty}^{e_i - e_k} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, t) dt \right] = \\ &= \int_{-\infty}^{e_i - e_k} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(e_i - e_k, t) dt + \int_{-\infty}^{e_i - e_j} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, e_i - e_k) ds \end{aligned} \tag{10}$$

By analogy, differentiating (6), we can find $\frac{\partial P_3}{\partial e_i}$:

$$\begin{aligned} \frac{\partial P_3}{\partial e_i} &= - \int_{e_i - e_k}^{\infty} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(e_i - e_k, t) dt - \int_{e_i - e_j}^{\infty} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, e_i - e_k) ds = \\ &= - \left(\int_{e_i - e_k}^{\infty} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(e_i - e_k, t) dt + \int_{e_i - e_j}^{\infty} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, e_i - e_k) ds \right) \end{aligned} \tag{11}$$

If the Cournot-Nash assumptions are satisfied, as every agent has symmetric reaction function, then in equilibrium each agent will choose the same optimal level of effort ex-ante, meaning that $e_i = e_j = e_k = e^*$ in equilibrium and thus ex-post probabilities of ranking first, second and third will be equal $P_1 = P_2 = P_3 = \frac{1}{3}$. Substituting these conditions into (10), due to symmetry of the distribution around 0 we get the following $\frac{\partial P_1}{\partial e_i}$:

$$\begin{aligned} \frac{\partial P_1}{\partial e_i} &= \int_{-\infty}^{0} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(0, t) dt + \int_{-\infty}^{0} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(s, 0) ds = \\ &= \int_{-\infty}^{\infty} f_{\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_k}(0, t) dt = \\ &= g(0) \end{aligned} \tag{12}$$

where g(.) is the marginal density function of the $\epsilon_i - \epsilon_j$, which under the assumption of normal distribution for $\epsilon_i - \epsilon_j$ with mean 0 and variance of $2\sigma_(\epsilon)^2$ becomes:

$$g(e_i - e_j) = \frac{1}{\sqrt{2}\sigma_\epsilon\sqrt{2\pi}} e^{\frac{-(e_i-e_j)^2}{2\sigma_\epsilon^2}} \tag{13}$$

Evaluated at 0:

$$g(0) = \frac{1}{\sqrt{2}\sigma_\epsilon\sqrt{2\pi}} = \frac{1}{2\sqrt{\pi\sigma_\epsilon^2}} \tag{14}$$

It is important to stress here that the above result will only hold for symmetric distributions with mean 0. This topic is further explored in great detail in Akerlof and Holden (2012).

By analogy with (12), in equillibrium $\frac{\partial P_3}{\partial e_i} = -g(0)$, so $\frac{\partial P_1}{\partial e_i} + \frac{\partial P_3}{\partial e_i} = 0$. It will be worth noting, that this result will hold in a more general form for symmetric distributions with mean 0: if the number of agents will be even, $2k$, then for all pairs of marginal probabilities for i from 1 to $2k$ the following will hold $\frac{\partial P_i}{\partial e_i} + \frac{\partial P_{i+2k-1}}{\partial e_i} = 0$, in case the number of agents will be odd, $2k+1$, marginal probability with respect to effort of getting exactly in the middle, i.e. ranking $k+1$, will be zero as in the case with 3 agents.

Substituting the above equilibrium conditions (12) and (14) into (9), we get the optimal effort as function of W_1 and W_3:

$$\begin{aligned}(W_1 - W_3)\frac{\partial P_1}{\partial e_i} - \frac{e_i}{a} &= 0 \Leftrightarrow \\ g(0)(W_1 - W_3) - \frac{e_i}{a} &= 0 \Leftrightarrow \\ \frac{1}{2\sqrt{\pi\sigma_\epsilon^2}}(W_1 - W_3) - \frac{e_i}{a} &= 0 \Leftrightarrow \\ e^* &= \frac{a}{2\sqrt{\pi\sigma_\epsilon^2}}(W_1 - W_3) \end{aligned} \tag{IC.1}$$

What this says is that when choosing an effort level to apply, the agent only considers the spread between the first and the last prize! Intermediate prize, W_2, doesn't appear in the optimal effort function at all. So when agents are risk neutral, the principal can elicit higher effort by increasing the spread between the first and the third price. This is by far the most interesting result of the paper, it is consistent with the finding of Lazear and Rosen (1981) for the two agents, although, this paper has used entirely different approach and did not restrict itself to the case of principal facing perfect competition in the market.

This result is far more intriguing though than the same one for two agents. And this is why: Lazear and Rosen found that the spread between W_1 and W_2 mattered for two agents engaging in the production, so all prized in the compensation scheme matter, but what this paper finds is that with greater number of contestants, three, not all of the prizes matter for the decision on how hard to work! The intuition might be as follows: when working at the large corporation you always think of the CEO's compensation as compared to yours, this difference is exactly the thing that motivates you to apply best effort in the whole course of you career.

2.d Individual Rationality

Principal knows that at the same time as being incentive-compatible, the expected prize must also translate into the expected utility at least equal to the agent's reservation utility $\overline{u}$, or the agent will simply choose not to work, in other words it has to satisfy individual rationality constraint:

$$EU \geq \overline{u}$$

Following Marinakis and Tsoulouhas (2013), I further assume that the principal has all the bargaining power and wants to insure participation at the least costs, so the agent is left with no rents and individual rationality constraint holds with an equality:

$$\begin{aligned} EU &= \overline{u} \Leftrightarrow \\ W_1P_1 + W_2P_2 + W_3P_3 - \frac{e_i^2}{2a} &= \overline{u} \end{aligned} \tag{15}$$

Substituting e^* from (IC.1) and equilibrium probabilities, further rearranging, we get an expression for W_2:

$$\begin{aligned} \frac{1}{3}(W_1+W_2+W_3) - \frac{\left(\frac{a}{2\sqrt{\pi\sigma_\epsilon^2}}(W_1-W_3)\right)^2}{2a} &= \overline{u} \Leftrightarrow \\ \frac{1}{3}(W_1+W_2+W_3) - \frac{a}{8\pi\sigma_\epsilon^2}(W_1-W_3)^2 &= \overline{u} \Leftrightarrow \\ \frac{1}{3}W_2 = \overline{u} + \frac{a}{8\pi\sigma_\epsilon^2}(W_1-W_3)^2 - \frac{1}{3}(W_1+W_3) &\Leftrightarrow \\ W_2 = 3\overline{u} + \frac{3a}{8\pi\sigma_\epsilon^2}(W_1-W_3)^2 - (W_1+W_3) & \end{aligned} \tag{IR.1}$$

2.e Profit Maximization

Finally, at stage 1 of the game, the principal maximizes her total expected profit subject to incentive compatibility (IC.1) and individual rationality (IR.1) constrains to find the prize structure that will ensure both participation and required level of effort from the agent and maximum profit for the principal:

$$\begin{aligned} E\Pi &= Ex_i + Ex_j + Ex_k - W_1 - W_2 - W_3 = \\ &= 3(e^* + a) - W_1 - W_2 - W_3 \end{aligned}$$

Substituting e^* from (IC.1) and W_2 from (IR.1):

$$\begin{aligned} E\Pi &= \frac{3a}{2\sqrt{\pi\sigma_\epsilon^2}}(W_1-W_3) + 3a - W_1 - \left(3\overline{u} + \frac{3a}{8\pi\sigma_\epsilon^2}(W_1-W_3)^2 - (W_1+W_3)\right) - W_3 = \\ &= \frac{3a}{2\sqrt{\pi\sigma_\epsilon^2}}(W_1-W_3) + 3a - W_1 - 3\overline{u} - \frac{3a}{8\pi\sigma_\epsilon^2}(W_1-W_3)^2 + W_1 + W_3 - W_3 = \\ &= \frac{3a}{2\sqrt{\pi\sigma_\epsilon^2}}(W_1-W_3) + 3a - 3\overline{u} - \frac{3a}{8\pi\sigma_\epsilon^2}(W_1-W_3)^2 \end{aligned} \tag{16}$$

The principal maximizes the expected profit with respect to $W_1 - W_3$. First order condition gives:

$$\frac{\partial E\Pi}{\partial(W_1 - W_3)} = 0 \Leftrightarrow$$
$$\frac{3a}{2\sqrt{\pi\sigma_\epsilon^2}} - \frac{6a}{8\pi\sigma_\epsilon^2}(W_1 - W_3) = 0 \Leftrightarrow$$
$$W_1 - W_3 = \frac{\frac{3a}{2\sqrt{\pi\sigma_\epsilon^2}}}{\frac{3a}{4\pi\sigma_\epsilon^2}} \Leftrightarrow$$
$$W_1 - W_3 = 2\sqrt{\pi\sigma_\epsilon^2} \tag{17}$$

2.f Solutions

This subsection will be devoted to finding the closed form solutions for the prizes. Any precise solutions for specific functions are entirely absent in Lazear and Rosen (1981) and others, including Akerlof and Holden (2012), because the researches tend to solve most general models, do not specify any distribution for individual utility and cost of effort functions, often use approximations and simulations, while this paper has an advantage of showing and analyzing results explicitly using concrete functional forms for the utility and cost of effort.

Substituting the spread into (IC.1) we get optimal effort:

$$e^* = \frac{a}{2\sqrt{\pi\sigma_\epsilon^2}}(W_1 - W_3) = \frac{a}{2\sqrt{\pi\sigma_\epsilon^2}}2\sqrt{\pi\sigma_\epsilon^2} = a \tag{18}$$

This is a new result, that apparently can be attributed to this paper: when agents are risk-neutral and homogeneous, the optimal level of effort for each agent is equal to her ability.

Substituting $W_1 - W_3$ from (17) back into (IR.1), we get a relationship between W_1 and W_2:

$$\begin{aligned} W_2 &= 3\overline{u} + \frac{3a}{8\pi\sigma_\epsilon^2}(2\sqrt{\pi\sigma_\epsilon^2})^2 - (W_1 + W_1 - 2\sqrt{\pi\sigma_\epsilon^2}) = \\ &= 3\overline{u} + \frac{3a}{8\pi\sigma_\epsilon^2}4\pi\sigma_\epsilon^2 - 2(W_1 - \sqrt{\pi\sigma_\epsilon^2}) = \\ &= 3\overline{u} + \frac{3a}{2} - 2(W_1 - \sqrt{\pi\sigma_\epsilon^2}) \end{aligned} \tag{19}$$

In the setup we specified that $W_2 = \alpha W_1 + (1-\alpha)W_3$, where $0 \leq \alpha \leq 1$, so from (17) and (18), we get the following system of equations:

$$\begin{cases} W_3 = W_1 - 2\sqrt{\pi\sigma_\epsilon^2} \\ \alpha W_1 + (1-\alpha)W_3 = 3\overline{u} + \frac{3a}{2} - 2(W_1 - \sqrt{\pi\sigma_\epsilon^2}) \end{cases}$$

$$(1-\alpha)W_3 = 3\overline{u} + \frac{3a}{2} - 2(W_1 - \sqrt{\pi\sigma_\epsilon^2}) - \alpha W_1 \Rightarrow$$
$$(1-\alpha)W_3 = 3\overline{u} + \frac{3a}{2} + 2\sqrt{\pi\sigma_\epsilon^2} - (2+\alpha)W_1 \Rightarrow$$
$$W_3 = \frac{3\overline{u} + \frac{3a}{2} + 2\sqrt{\pi\sigma_\epsilon^2} - (2+\alpha)W_1}{(1-\alpha)}$$

So W_1 is:

$$W_1 - 2\sqrt{\pi\sigma_\epsilon^2} = \frac{3\overline{u} + \frac{3a}{2} + 2\sqrt{\pi\sigma_\epsilon^2} - (2+\alpha)W_1}{(1-\alpha)} \Rightarrow$$
$$(1-\alpha)(W_1 - 2\sqrt{\pi\sigma_\epsilon^2}) = 3\overline{u} + \frac{3a}{2} + 2\sqrt{\pi\sigma_\epsilon^2} - (2+\alpha)W_1 \Rightarrow$$
$$(1-\alpha+2+\alpha)W_1 = (1-\alpha)2\sqrt{\pi\sigma_\epsilon^2} + 3\overline{u} + \frac{3a}{2} + 2\sqrt{\pi\sigma_\epsilon^2} \Rightarrow$$
$$3W_1 = (2-\alpha)2\sqrt{\pi\sigma_\epsilon^2} + 3\overline{u} + \frac{3a}{2} \Rightarrow$$

$$\boxed{W_1 = \frac{2}{3}(2-\alpha)\sqrt{\pi\sigma_\epsilon^2} + \overline{u} + \frac{a}{2}} \tag{20}$$

So W_3 is:

$$W_3 = \frac{2}{3}(2-\alpha)\sqrt{\pi\sigma_\epsilon^2} + \overline{u} + \frac{a}{2} - 2\sqrt{\pi\sigma_\epsilon^2} \Rightarrow$$
$$\boxed{W_3 = -\frac{2}{3}(1+\alpha)\sqrt{\pi\sigma_\epsilon^2} + \overline{u} + \frac{a}{2}} \tag{21}$$

And W_2 is:

$$W_2 = \alpha 2\sqrt{\pi\sigma_\epsilon^2} - \frac{2}{3}(1+\alpha)\sqrt{\pi\sigma_\epsilon^2} + \overline{u} + \frac{a}{2} \Rightarrow$$
$$\boxed{W_2 = \frac{2}{3}(2\alpha-1)\sqrt{\pi\sigma_\epsilon^2} + \overline{u} + \frac{a}{2}} \tag{22}$$

Summing (19), (20) and (21), we find that the total payment principal makes to agents is independent of α, it does not matter neither for the agents nor for the principal whether W_2 is set equal to W_1 or W_3. If the principal set $\alpha = 0$, the resulted prizes would be $W_1 = \frac{4}{3}\sqrt{\pi\sigma_\epsilon^2} + \overline{u} + \frac{a}{2}$, $W_2 = W_3 = -\frac{2}{3}\sqrt{\pi\sigma_\epsilon^2} + \overline{u} + \frac{a}{2}$. We can see that each prize includes a base payment, consisting of agent's reservation utility and some compensation for the effort (equal to ability), and a "fee" increasing in volatility of production. This results is exactly the one described by Lazear and Rosen (1981), so we can conclude that their findings are robust to the drop of zero-profit assumption.

Summarizing the findings of this section, we see that in the game with three homogeneous risk-neutral agents, all agents choose to apply the same effort, only taking into account the spread between the first and the last price, this optimal efforts turns out to be equal the agent's ability. The setting does not allow to identify the unique structure of the prizes, but a continuum of such structures. Taking one numerical example we saw, that in this kind of game the optimal compensation can be viewed as consisting of the basic payment and a bonus or penalty depending on the rank, which reminds as of the cardinal tournament compensation scheme.

3 The Rank-Order Tournament with 3 Risk-Averse Agents

3.a Setup

In this section quite unrealistic assumption that agents are rick-neutral is revised and instead agents are assumed to be risk-averse. For the analysis of this case a widely used in literature (e.g. in Lazear and Rosen (1981), Marinakis and Tsoulouhas (2013)), so called CARA utility function, function that posses constant absolute risk aversion, is chosen, so the utility for agent i is now:

$$u(w_i, e_i) = -exp\left(-r\left(w_i - \frac{e_i^2}{2a}\right)\right) \tag{23}$$

The rest of the setup duplicates that of 2.a.

3.b Probabilities

This section does not change from 2.b, as the only modified assumption is the utility function and the form of the utility function does not affect probabilities, so P_1, P_2 and P_3 are identical to 2.b.

3.c Incentive Compatibility

Just as in case with risk-neutral players, the principal needs to find the expected utility for agent i, which is a little more complicated now:

$$\begin{aligned} EU = -exp\left(-r\left(W_1 - \frac{e_i^2}{2a}\right)\right)P_1 - exp\left(-r\left(W_2 - \frac{e_i^2}{2a}\right)\right)P_2 - exp\left(-r\left(W_3 - \frac{e_i^2}{2a}\right)\right)P_3 = \\ = -exp\left(-r\left(W_1 - \frac{e_i^2}{2a}\right)\right)P_1 - exp\left(-r\left(W_2 - \frac{e_i^2}{2a}\right)\right)(1 - P_1 - P_3) - exp\left(-r\left(W_3 - \frac{e_i^2}{2a}\right)\right)P_3 \end{aligned} \tag{24}$$

The principal maximizes the agent's expected utility (24) with respect to e_i to find the optimal effort for any values of W_1, W_2 and W_3. First order condition is:

$$\frac{\partial EU}{\partial e_i} = 0$$

$$\Leftrightarrow$$

$$\begin{aligned} &- exp\left(-r\left(W_1 - \frac{e_i^2}{2a}\right)\right)\left(\frac{r}{a}e_i\right)P_1 - exp\left(-r\left(W_1 - \frac{e_i^2}{2a}\right)\right)\frac{\partial P_1}{\partial e_i} - \\ &- exp\left(-r\left(W_2 - \frac{e_i^2}{2a}\right)\right)\left(\frac{r}{a}e_i\right)(1 - P_1 - P_3) - exp\left(-r\left(W_2 - \frac{e_i^2}{2a}\right)\right)(-\frac{\partial P_1}{\partial e_i} - \frac{\partial P_3}{\partial e_i}) - \\ &- exp\left(-r\left(W_3 - \frac{e_i^2}{2a}\right)\right)\left(\frac{r}{a}e_i\right)P_3 - exp\left(-r\left(W_3 - \frac{e_i^2}{2a}\right)\right)\frac{\partial P_3}{\partial e_i} = 0 \end{aligned}$$

$$\Leftrightarrow$$

$$-\,exp\left(-r\left(W_1-\frac{e_i^2}{2a}\right)\right)\left[\frac{r}{a}e_iP_1+\frac{\partial P_1}{\partial e_i}\right]-$$
$$-\,exp\left(-r\left(W_2-\frac{e_i^2}{2a}\right)\right)\left[\frac{r}{a}e_i(1-P_1-P_3)-\left(\frac{\partial P_1}{\partial e_i}+\frac{\partial P_3}{\partial e_i}\right)\right]- \tag{25}$$
$$-\,exp\left(-r\left(W_3-\frac{e_i^2}{2a}\right)\right)\left[\frac{r}{a}e_iP_3+\frac{\partial P_3}{\partial e_i}\right]=0$$

Just as in the case with risk-neutral agents, if the Cournot-Nash assumptions are satisfied, then $e_i = e_j = e_k = e^*$ in equilibrium and thus $P_1 = P_2 = P_3 = \frac{1}{3}$. In equilibrium $\frac{\partial P_1}{\partial e_i} = -\frac{\partial P_3}{\partial e_i}$. Substituting equilibrium conditions and (12)-(14) into (25), we get the optimal effort:

$$-\,exp\left(-r\left(W_1-\frac{e_i^2}{2a}\right)\right)\left[\frac{r}{a}e_i\frac{1}{3}+\frac{\partial P_1}{\partial e_i}\right]-$$
$$-\,exp\left(-r\left(W_2-\frac{e_i^2}{2a}\right)\right)\left[\frac{r}{a}e_i\frac{1}{3}\right]-$$
$$-\,exp\left(-r\left(W_3-\frac{e_i^2}{2a}\right)\right)\left[\frac{r}{a}e_i\frac{1}{3}-\frac{\partial P_1}{\partial e_i}\right]=0$$
$$\Leftrightarrow$$
$$-exp\left(\frac{r}{2a}e_i^2\right)\left[exp\left(-rW_1\right)\left[\frac{r}{3a}e_i+\frac{\partial P_1}{\partial e_i}\right]+exp\left(-rW_2\right)\left[\frac{r}{3a}e_i\right]+exp\left(-rW_3\right)\left[\frac{r}{3a}e_i-\frac{\partial P_1}{\partial e_i}\right]\right]=0$$
$$\Leftrightarrow$$
$$exp\left(-rW_1\right)\left[\frac{r}{3a}e_i+\frac{\partial P_1}{\partial e_i}\right]+exp\left(-rW_2\right)\left[\frac{r}{3a}e_i\right]+exp\left(-rW_3\right)\left[\frac{r}{3a}e_i-\frac{\partial P_1}{\partial e_i}\right]=0$$
$$\Leftrightarrow$$
$$\frac{r}{3a}e_i\left(exp\left(-rW_1\right)+exp\left(-rW_2\right)+exp\left(-rW_3\right)\right)+\frac{\partial P_1}{\partial e_i}\left(exp\left(-rW_1\right)-exp\left(-rW_3\right)\right)=0$$
$$\Leftrightarrow$$
$$\frac{r}{3a}e_i\left(exp\left(-rW_1\right)+exp\left(-rW_2\right)+exp\left(-rW_3\right)\right)=-g(0)\left(exp\left(-rW_1\right)-exp\left(-rW_3\right)\right)$$
$$\Leftrightarrow$$
$$\frac{r}{3a}e_i=-\frac{g(0)\left(exp\left(-rW_1\right)-exp\left(-rW_3\right)\right)}{exp\left(-rW_1\right)+exp\left(-rW_2\right)+exp\left(-rW_3\right)}$$
$$\Leftrightarrow$$
$$e^*=-\frac{3a}{r}\frac{g(0)\left(exp\left(-rW_1\right)-exp\left(-rW_3\right)\right)}{exp\left(-rW_1\right)+exp\left(-rW_2\right)+exp\left(-rW_3\right)}$$
$$\Leftrightarrow$$
$$e^*=-\frac{3a}{2r\sqrt{\pi\sigma_\epsilon^2}}\frac{exp\left(-rW_1\right)-exp\left(-rW_3\right)}{exp\left(-rW_1\right)+exp\left(-rW_2\right)+exp\left(-rW_3\right)} \tag{IC.2}$$

This incentive-compatibility constraint tells us that, in contrast with a case of risk-neutral agents, the intermediate prize now does matter for the risk-averse agent when she chooses how hard to work her whole life. Again, despite the different approaches, this result is consistent with analysis for just two agents in Lazear and Rosen (1981). This result in itself is important because, although Lazear and Rosen (1981) find that for two risk-averse agents it is the wages themselves, not the spread matter, it was not clear whether the second prize would have influence on agent's decision for 3 agents or not. This paper finds that it does. The intuition behind it may be that under risk-aversion, the expected payment is not equal to the expected utility of that payment and agents prefer a more certain income.

3.d Individual Rationality

As in the case with risk-neutral players, compensation scheme must satisfy individual rationality constraint, i.e. result in the expected utility no less than agent's reservation utility:

$$EU \geq \overline{u}$$

Again, I assume that the principal has all the bargaining power and wants to insure participation at the least costs, so the agent is left with no rents and individual rationality constraint holds with an equality:

$$EU = \overline{u} \Leftrightarrow$$

$$EU = -exp\left(-r\left(W_1 - \frac{e_i^2}{2a}\right)\right)P_1 - exp\left(-r\left(W_2 - \frac{e_i^2}{2a}\right)\right)P_2 - exp\left(-r\left(W_3 - \frac{e_i^2}{2a}\right)\right)P_3 = \overline{u}$$

$$\Leftrightarrow$$

$$-exp\left(\frac{r}{2a}e_i^2\right)\left(exp\left(-rW_1\right)P_1 + exp\left(-rW_2\right)P_2 + exp\left(-rW_3\right)P_3\right) = \overline{u}$$

$$\Leftrightarrow$$

$$exp\left(-rW_1\right)P_1 + exp\left(-rW_2\right)P_2 + exp\left(-rW_3\right)P_3 = -\frac{\overline{u}}{exp\left(\frac{r}{2a}e_i^2\right)}$$

$$\Leftrightarrow$$

$$exp\left(-rW_1\right)\frac{1}{3} + exp\left(-rW_2\right)\frac{1}{3} + exp\left(-rW_3\right)\frac{1}{3} = -\frac{\overline{u}}{exp\left(\frac{r}{2a}e_i^2\right)}$$

$$\Leftrightarrow$$

$$exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right) = -\frac{3\overline{u}}{exp\left(\frac{r}{2a}e_i^2\right)} \tag{IR.2}$$

Substituting e^* from (IC.2) into (IR.2), we have the following equation:

$$exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right) = -\frac{3\overline{u}}{exp\left(\frac{r}{2a}\left(-\frac{3a}{2r\sqrt{\pi\sigma_\epsilon^2}}\frac{exp(-rW_1)-exp(-rW_3)}{exp(-rW_1)+exp(-rW_2)+exp(-rW_3)}\right)^2\right)} =$$

$$= -\frac{3\overline{u}}{exp\left(\frac{9a}{8r\pi\sigma_\epsilon^2}\frac{(exp(-rW_1)-exp(-rW_3))^2}{(exp(-rW_1)+exp(-rW_2)+exp(-rW_3))^2}\right)} \tag{26}$$

3.e Profit Maximization

As shown in the Appendix 2, there is no closed-form explicit solution for the profit maximization problem for the case with 3 risk-averse agents.

3.f Results

Although the case when the principal signs a contract with three risk-averse agents, instead of risk-neutral, does not have an explicit solution for the chosen utility function and for the simpler one (see

Appendix 1), we can gain some insights by using advanced optimization techniques that will give as approximate optimal values of W_1, W_2 and W_3, for different values of parameters we get the results presented in Table 1.

Table 1. Constant Absolute Risk Aversion

Ability (a)	Risk-Aversion (r)	W_1	W_2	W_3
$\sigma_\epsilon^2 = 0.1$				
5	1	158.2426	-303.8109	-353.2793
5	2	135.7474	-143.3160	-176.8977
10	1	160.6809	-306.2753	-353.2221
10	2	134.5159	-142.2186	-176.6396
$\sigma_\epsilon^2 = 0.5$				
Ability (a)	Risk-Aversion (r)	W_1	W_2	W_3
5	1	145.8577	-291.5832	-354.2535
5	2	134.1776	-142.9689	-177.1268
10	1	159.1269	-305.3333	-353.9070
10	2	134.1776	-142.9689	-177.1268
$\sigma_\epsilon^2 = 1$				
Ability (a)	Risk-Aversion (r)	W_1	W_2	W_3
5	1	147.3952	-295.0908	-354.6001
5	2	131.0489	-139.6012	-177.4457
10	1	145.8577	-291.5832	-354.2535
10	2	139.1595	-144.5657	-177.3001

Everywhere in the Table 1 $\overline{u} = -1$, starting values for W_1, W_2 and W_3 are [100 100 100].

Table 1 shows an interesting result: for different values of ability, risk-aversion and production volatility, coming from idiosyncratic risks, prize for the first place is always positive, but the prizes for second and third are actually negative, usually an agents pays a greater sum for not winning and even greater for loosing, than the sum she gets for winning the tournament!

To summarize, there are two main learnings from the work on this section. The first one is that when agents are averse towards risk, their choice of the optimal level of effort does depend on all the prizes W_1, W_2 and W_3, not just the spread between the first and the last prize. This does not contradict Lazear and Rosen (1981). And the second is that optimal wages structure suggest that winner gets positive prize, while looser have to pay for their loss.

4 The Rank-Order Tournament with N Risk-Neutral Agents

4.a Setup

In their prominent paper Lazear and Rosen (1981) describe a tournament for just 2 players, while this paper went further and attempted to analyze ordinal tournament with 3 contestants, main challenge was to confirm findings for 2-3 players taking N agents. Malcomson (1986) took the case of a continuum of agents to compare piece rates and rank-order tournaments, but focused on the performance of agents not being public information affecting the relative optimum of the two schemes. Just lately, Akerlof and Holden (2012) performed an extensive analysis of this case, though not providing a wage structure specifically. This section attempts to do a similar analysis, assuming though that agents are discrete, and describes difficulties that arise. So taking a simpler case with risk-neutral players, the setup here is the same as in 2.a, but the number of agents is N.

Under the ordinal tournament the wage received by the agent i depends on the rank of x_i:

$$w_i = \begin{cases} W_1 & \text{if } x_i > x_j \; \forall i \neq j \\ W_2 & \\ \vdots & \\ W_n & \text{if } x_i < x_j \; \forall i \neq j \end{cases} \tag{27}$$

where $W_1 \geq W_2 \geq ... \geq W_n$.

4.b Probabilities

The principal wants to find the the expected utility and maximize it to find the effort level that the agent i will chooses.

$$\begin{aligned} EU &= W_1 P_1 + W_2 P_2 + ... + W_n P_n - \frac{e_i^2}{2a} = \\ &= \sum_{s=1}^{n} W_s P_s - \frac{e_i^2}{2a} \end{aligned} \tag{28}$$

To calculate the expected utility for the agent i we need the probability of each payoff:

$$P(\text{Rank First}) = P_1 = P(x_i > x_j \; and \; x_i > x_k \; and \; x_i > x_f, \; etc)$$

$$\vdots$$

$$P(\text{Rank N}) = P_n = P(x_i < x_j \; and \; x_i < x_k \; and \; x_i < x_f, \; etc)$$

Unfortunately, there exist no compact representations for this probabilities:

$$P_1 = F_{\epsilon_i - \epsilon_j, \ldots, \epsilon_i - \epsilon_k}(e_i - e_j, \ldots, e_i - e_k) =$$

$$= \underbrace{\int_{-\infty}^{e_i - e_j} \cdots \int_{-\infty}^{e_i - e_k}}_{\text{n times}} f_{\epsilon_i - \epsilon_j, \ldots, \epsilon_i - \epsilon_k}(s, \ldots, t) ds, \ldots, dt \tag{29}$$

Obviously, probabilities must sum up to 1:

$$P_1 + P_2 + ... + P_n = 1 \Rightarrow$$
$$\sum_{s=1}^{n} P_s = 1 \tag{30}$$

Taking full differential of (30) wrt e_i, we get:

$$\frac{\partial P_1}{\partial e_i} + \frac{\partial P_2}{\partial e_i} + ... + \frac{\partial P_n}{\partial e_i} = 0 \Rightarrow$$
$$\sum_{s=1}^{n} \frac{\partial P_s}{\partial e_i} = 0 \tag{31}$$

As discussed in more detail in 2.c, the relationship among these marginal probabilities is as follows depends on symmetry on the underlying distribution and whether the number of agents is odd or even.

4.c Incentive Compatibility

To ensure that the compensation scheme is incentive-compatible, the principal maximizes expected utility of each agent (28) with respect to effort level e_i. First order condition is:

$$\frac{\partial EU}{\partial e_i} = 0 \;\; \Leftrightarrow$$
$$W_1 \frac{\partial P_1}{\partial e_i} + W_2 \frac{\partial P_2}{\partial e_i} + ... + W_n \frac{\partial P_n}{\partial e_i} - \frac{e_i}{a} = 0 \tag{32}$$

If the Cournot-Nash assumptions are satisfied, then $e_i = e_j = e_k = e^*$ in equilibrium and thus ex-post probabilities are equal in equilibrium, i.e. $P_1 = P_2 = ... = P_n = \frac{1}{n}$.

With N players, we can not explicitly derive values of marginal probabilities in a form we derived then in 2.c, nor can we understand the relationship between them beyond the point already mentioned. This is the main obstacle on our way in this section. We can not conclude whether optimal effort depends only on the spread between the first and the last prize, or on each prize in the scheme:

$$e^* = a(W_1 \frac{\partial P_1}{\partial e_i} + W_2 \frac{\partial P_2}{\partial e_i} + ... + W_n \frac{\partial P_n}{\partial e_i}) \tag{IC.3}$$

4.d Individual Rationality

To satisfy individual rationality constraint, the principal also needs to take into account that the optimal compensation scheme must ensure expected utility (28) at least equal to agent's reservation utility $\overline{u}$:

$$EU \geq \overline{u}$$

Once again, I assume that the principal has all the bargaining power and wants to insure participation at the least costs, so the agent is left with no rents and individual rationality constraint holds with an equality:

$$EU = \overline{u}$$
$$\Leftrightarrow$$
$$\sum_{s=1}^{n} W_s P_s - \frac{e_i^2}{2a} = \overline{u} \tag{33}$$

Substituting equilibrium conditions and e^* from (IC.3), we get the following expression:

$$\frac{1}{n}\sum_{s=1}^{n} W_s - \frac{(a(W_1\frac{\partial P_1}{\partial e_i} + W_2\frac{\partial P_2}{\partial e_i} + ... + W_n\frac{\partial P_n}{\partial e_i}))^2}{2a} = \overline{u}$$
$$\Leftrightarrow$$
$$\sum_{s=1}^{n} W_s = \frac{an}{2}((W_1\frac{\partial P_1}{\partial e_i} + W_2\frac{\partial P_2}{\partial e_i} + ... + W_n\frac{\partial P_n}{\partial e_i}))^2 + n\overline{u} \tag{IR.3}$$

4.e Profit Maximization

The principal's expected profit is again:

$$\begin{aligned} E\Pi = Ex_i + \ldots + Ex_k - \sum_{s=1}^{n} W_s = \\ = n(e^* + a) - \sum_{s=1}^{n} W_s \end{aligned} \tag{34}$$

Substituting e^* from (IC.3) and $\sum_{s=1}^{n} W_s$ from (IR.3), the expression for expected profit becomes:

$$E\Pi = na(W_1\frac{\partial P_1}{\partial e_i} + W_2\frac{\partial P_2}{\partial e_i} + ... + W_n\frac{\partial P_n}{\partial e_i}) + na - \frac{an}{2}((W_1\frac{\partial P_1}{\partial e_i} + W_2\frac{\partial P_2}{\partial e_i} + ... + W_n\frac{\partial P_n}{\partial e_i}))^2 + n\overline{u} \tag{35}$$

Maximization of total expected profit (35) with respect to a set of wages $W_1,\ldots,W_n$ does not help us determine the optimal compensation scheme.

4.f Results

The objective of this section was to see whether a striking result that risk-neutral players take into account only the spread between the first and the last prize that was discovered by Lazear and Rosen (1981) for the game with two players and confirmed for the one with three in 2.c of this paper, will hold for more general setting with N agents. It appears from the above analyzes that the answer is not clear and depends on the relationship between the marginal probabilities.

5 Conclusions

This paper was based mainly on the famous work of Lazear and Rosen (1981), that effectively established the theory of rank-order tournaments. It was developed in parallel with more recent work of Akerlof and Holden (2012), though having slightly different objectives.

Lazear and Rosen started their article by stating that: "It is a familiar proposition that under competitive conditions workers are paid the value of their marginal products." So the assumption of perfect competition is vital to their work. What this research had as its purpose was relaxing the assumption of perfect competition and letting the principal enjoy economic profits. This led to view of the tournament as a non-cooperative game and required choosing backwards induction as the method for solving it. In this regard the main finding is that imposition of a perfect competition in the market for the agent's output was not crucial to the validity of results: all of the findings are consistent with a view of Lazear and Rosen (1981).

Another objective of this research was obtaining closed-form solutions for the prize-structure using specific utility and cost of effort functions for the agents, instead of the general form and Taylor approximations used in the original paper. It turned out that it was only possible to so for the case with three risk-neutral agents, found solutions provide some further insights into the theory of tournaments and will be discussed in more detail below.

Next area for this research was proving that results that were shown by Lazear and Rosen (1981) for the contract with two risk-neutral agents would hold for three. The finding that in the ordinal tournament with two risk-neutral players when choosing the optimum level of effort agents only take into consideration the spread between the first and the last prize is confirmed in this paper for the game with three risk-neutral agents and relaxed zero-profit assumption inherent to the analysis of Lazear and Rosen (1981) and some of their followers. This finding is important because Lazear and Rosen basically stated that when there two prizes, the spread between them matter, but what if there were more prizes? Would it be spreads between successive prizes that matters? For three players it appear that only the spread between the first and last one matters. Moreover, another result that deserves attention and seems unique to this paper and was not possible to notice in Lazear and Rosen (1981), because the production function in their research did not include ability of the agent, is that the agent chooses the optimal effort level simply equal to her ability.

Although it was not possible to find an explicit solution to the game with three risk-averse agents, the main result of the case when agents care for risks is that the optimum level of effort they choose now definitely depends on all the prizes values, not just the spread between first and the last one as in the case with risk-neutral agents. This is again in line with analysis by Lazear and Rosen (1981).

The last goal of this work was to see whether a striking result that risk-neutral players take into account only the spread between the first and the last prize that was discovered by Lazear and Rosen (1981) for the game with two players and confirmed for the one with three in 2.c of this paper, will hold for more general setting with N agents. It appears from the above analyzes that the answer is not clear and depends on the relationship between the marginal probabilities.

Summarizing it all, this paper confirmed most of the results of Lazear and Rosen (1981) it chose for exploring along with finding some new and facing obstacles not mentioned in the original paper, but also described in detail on Akerlof and Holden (2012).

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Appendices

A Appendix 1

Another function popular in literature is a utility function, characterized by declining absolute risk aversion (DARA):

$$u(w_i, e_i) = r(w_i - \frac{e_i^2}{2})^r \tag{36}$$

The expected utility for agent i in the tournament for 3 agents under the above specification is:

$$EU = r(W_1 - \frac{e_i^2}{2})^r P_1 + r(W_2 - \frac{e_i^2}{2})^r P_2 + r(W_3 - \frac{e_i^2}{2})^r P_3 \tag{37}$$

Maximizing the expected utility for agent i (37) with respect to effort, e_i, we get the following first order condition:

$$\begin{aligned}\frac{\partial EU}{\partial e_i} =& - r^2 e_i P_1 \left(W_1 - \frac{e_i^2}{2}\right)^{(r-1)} + r\left(W_1 - \frac{e_i^2}{2}\right)^r \frac{\partial P_1}{\partial e_i} - \\ & - r^2 e_i P_2 \left(W_2 - \frac{e_i^2}{2}\right)^{(r-1)} + r\left(W_2 - \frac{e_i^2}{2}\right)^r \frac{\partial P_2}{\partial e_i} - \\ & - r^2 e_i P_3 \left(W_3 - \frac{e_i^2}{2}\right)^{(r-1)} + r\left(W_3 - \frac{e_i^2}{2}\right)^r \frac{\partial P_3}{\partial e_i} = 0\end{aligned} \tag{38}$$

Rearranging further:

$$\begin{aligned}\frac{\partial EU}{\partial e_i} =& \left(W_1 - \frac{e_i^2}{2}\right)^{(r-1)} (-r^2 e_i P_1 + r\left(W_1 - \frac{e_i^2}{2}\right) \frac{\partial P_1}{\partial e_i}) - \\ & \left(W_2 - \frac{e_i^2}{2}\right)^{(r-1)} (-r^2 e_i P_2 + r\left(W_2 - \frac{e_i^2}{2}\right) \frac{\partial P_2}{\partial e_i}) - \\ & \left(W_3 - \frac{e_i^2}{2}\right)^{(r-1)} (-r^2 e_i P_3 + r\left(W_3 - \frac{e_i^2}{2}\right) \frac{\partial P_3}{\partial e_i}) = 0\end{aligned} \tag{39}$$

So, from (39) we can clearly see that we will not be able to present an analytical solution for the optimal effort and thus it was decided not to proceed with this specification.

B Appendix 2

The principals expected profit is again:

$$\begin{aligned}E\Pi =& Ex_i + Ex_j + Ex_k - W_1 - W_2 - W_3 = \\ &= 3(e^* + a) - W_1 - W_2 - W_3\end{aligned}$$

Substituting e^* from (IC.2), the expression for expected profit becomes:

$$E\Pi = -\frac{9a}{2r\sqrt{\pi\sigma_\epsilon^2}}\frac{exp\left(-rW_1\right) - exp\left(-rW_3\right)}{exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right)} + 3a - W_1 - W_2 - W_3$$

Given (26) and (28), the principal faces the following constrained optimization problem now, whereby she maximizes profits subject to individual rationality constraint:

$$\begin{aligned}
&\underset{W_1,W_2,W_3}{\text{maximize}} && E\Pi = -\frac{9a}{2r\sqrt{\pi\sigma_\epsilon^2}}\frac{exp\left(-rW_1\right) - exp\left(-rW_3\right)}{exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right)} + 3a - W_1 - W_2 - W_3\\
&\text{subject to} && exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right) = -\frac{3\overline{u}}{exp\left(\frac{9a}{8r\pi\sigma_\epsilon^2}\frac{(exp(-rW_1)-exp(-rW_3))^2}{(exp(-rW_1)+exp(-rW_2)+exp(-rW_3))^2}\right)}
\end{aligned}$$

Which can be simplified to:

$$\underset{W_1,W_2,W_3}{\text{maximize}}\, E\Pi = -\frac{9a}{2r\sqrt{\pi\sigma_\epsilon^2}}\frac{exp\left(-rW_1\right) - exp\left(-rW_3\right)}{-\frac{3\overline{u}}{exp\left(\frac{9a}{8r\pi\sigma_\epsilon^2}\frac{(exp(-rW_1)-exp(-rW_3))^2}{(exp(-rW_1)+exp(-rW_2)+exp(-rW_3))^2}\right)}} + 3a - W_1 - W_2 - W_3$$

$$\Leftrightarrow$$

$$\underset{W_1,W_2,W_3}{\text{maximize}}\, E\Pi = \frac{3a}{2r\overline{u}\sqrt{\pi\sigma_\epsilon^2}}(exp\left(-rW_1\right) - exp\left(-rW_3\right))exp\left(\frac{9a}{8r\pi\sigma_\epsilon^2}\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) +$$
$$+3a - W_1 - W_2 - W_3$$

For the sake of manageability of calculations, let as introduce some simplifying notation $\frac{3a}{2r\overline{u}\sqrt{\pi\sigma_\epsilon^2}} = B$ and $\frac{9a}{8r\pi\sigma_\epsilon^2} = C$. Then the first of the three first order conditions looks like the following:

$$\begin{gathered}
\frac{\partial E\Pi}{\partial W_1} = -Brexp\left(-rW_1\right)exp\left(C\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) +\\
B(exp\left(-rW_1\right) - exp\left(-rW_3\right))exp\left(C\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right)\cdot\\
\cdot\left(2C\frac{-rexp\left(-rW_1\right)\left(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right)\right) + rexp\left(-rW_1\right)\left(exp\left(-rW_1\right) - exp\left(-rW_3\right)\right)}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) = 1
\end{gathered}$$

$$\Leftrightarrow$$

$$\begin{gathered}
\frac{\partial E\Pi}{\partial W_1} = -rexp\left(-rW_1\right) + \left(exp\left(-rW_1\right) - exp\left(-rW_3\right)\right)\left(2C\frac{-rexp\left(-rW_1\right)\left(exp\left(-rW_2\right) + 2exp\left(-rW_3\right)\right)}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) =\\
\frac{1}{Bexp\left(C\frac{(exp(-rW_1)-exp(-rW_3))^2}{(exp(-rW_1)+exp(-rW_2)+exp(-rW_3))^2}\right)}
\end{gathered} \tag{A.1}$$

Then the second of the three first order conditions is:

$$\frac{\partial E\Pi}{\partial W_2} = B(exp\left(-rW_1\right) - exp\left(-rW_3\right))exp\left(C\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) \cdot$$
$$\cdot \left(2rCexp\left(-rW_2\right)\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^3}\right) = 1$$
$$\Leftrightarrow$$
$$\frac{\partial E\Pi}{\partial W_2} = (exp\left(-rW_1\right) - exp\left(-rW_3\right))\left(2rCexp\left(-rW_2\right)\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^3}\right) =$$
$$= \frac{1}{Bexp\left(C\frac{(exp(-rW_1) - exp(-rW_3))^2}{(exp(-rW_1) + exp(-rW_2) + exp(-rW_3))^2}\right)} \tag{A.2}$$

Then the third one is:

$$\frac{\partial E\Pi}{\partial W_3} = Brexp\left(-rW_3\right)exp\left(C\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) +$$
$$B(exp\left(-rW_1\right) - exp\left(-rW_3\right))exp\left(C\frac{(exp\left(-rW_1\right) - exp\left(-rW_3\right))^2}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) \cdot$$
$$\cdot \left(2C\frac{rexp\left(-rW_3\right)\left(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right)\right) + rexp\left(-rW_3\right)\left(exp\left(-rW_1\right) - exp\left(-rW_3\right)\right)}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) = 1$$
$$\Leftrightarrow$$
$$\frac{\partial E\Pi}{\partial W_3} = rexp\left(-rW_3\right) + (exp\left(-rW_1\right) - exp\left(-rW_3\right))\left(2C\frac{rexp\left(-rW_3\right)\left(2exp\left(-rW_1\right) + exp\left(-rW_2\right)\right)}{(exp\left(-rW_1\right) + exp\left(-rW_2\right) + exp\left(-rW_3\right))^2}\right) =$$
$$\frac{1}{Bexp\left(C\frac{(exp(-rW_1) - exp(-rW_3))^2}{(exp(-rW_1) + exp(-rW_2) + exp(-rW_3))^2}\right)} \tag{A.3}$$

Thus we have a system on three non-linear equations with parameters (A.1), (A.2) and (A.3), for which it is impossible to find an analytical solution, so the solution is to choose reasonable values for the parameters and use advanced optimization methods in Matlab to gain insights into the relationship between optimal values of W_1, W_2 and W_3.